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Global Energy-Momentum Conservation in General Relativity

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It is shown that there exists a family of coordinate systems in which the energy-momentum tensor is globally conserved. Furthermore, this preferred class of frames includes geodesic systems with respect to any arbitrary point or timelike geodesic line. This implies a physically satisfactory conservation law with no need to introduce an extraneous pseudotensor.

1. INTRODUCTION

A new approach to the problem of energy and its conservation in the framework of general relativity has been recently proposed (Nissani and Leibowitz, 1988). The requirement that the energy-momentum be conserved plays a principal role in establishing the field equation of general relativity. It is, however, generally accepted that the covariant expression of the conservation law, i.e., the covariant divergencelessness condition imposed by Einstein's field equations

$$T^{\alpha\beta}_{;\beta} = 0 \tag{1}$$

(Greek indices run from 0 to 4, and semicolon denotes covariant derivative) leads only to a local continuity equation,

$$(\sqrt{-g}\, T^{\alpha\beta})_{,\beta} = 0 \tag{2}$$

valid in geodesic coordinates. Physically speaking, the local continuity equation is an expression of a conservation law in a very limited sense, holding strictly only in an infinitesimal space-time region.

The thrust of the efforts in the conventional approach has been to convert this local continuity equation, valid only in a preferred system of coordinates, into a global condition valid in all systems of coordinates. This

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aim is accomplished by adding an adequate complementary gravitational energy-momentum pseudotensor (Einstein, 1916; Rosen, 1940; Landau and Lifshitz, 1951; Moller, 1958; Goldberg, 1958; Komar, 1959; Cornish, 1964; Trautman, 1967; Penrose, 1982; Kovacs, 1985). Thereby a conservation law,

$$(\sqrt{-g}\; T^{\beta}_{\alpha} + t^{\beta}_{\alpha})_{,\beta} = 0 \tag{3}$$

is obtained, which is globally valid in all coordinate systems.

It is generally agreed that this approach has not led to satisfactory results (Maddox, 1985), suggesting renewed analysis of the fundamental assumptions underlying it. The goal to arrive at an equation of the type (3) derives from an implicit requirement of a separate general covariant conservation law for each component of the total energy-momentum. (Here and subsequently the term "general covariant" is used in the sense of being valid in all systems of coordinates.) We first question the need for such a requirement, which seems to entail nontensorial character of the gravitational energy-momentum [see, however, Nissani (1984)]. In fact, there is an intrinsic contradiction between "tensorial components conservation" and "general covariance," due to the mixing of the components in a "rotating" system. Furthermore, it is mathematically impossible for a non-antisymmetric second-order tensor to satisfy a general covariant continuity equation, namely, an ordinary divergencelessness condition in all coordinate systems.

In the already mentioned new approach a preferred class of coordinate systems is singled out. These systems are distinguished by the particular form the covariant divergencelessness condition (1) assumes in them. It transforms, when these special systems are utilized, into an ordinary divergencelessness condition valid through a finite region of spacetime. Thus, in these systems the energy-momentum tensor satisfies a global continuity equation in curved spacetime. Furthermore, there exists a nonempty intersection of this class with the set of locally geodesic coordinates relative to an arbitrary observer. The members of this intersection constitute a doubly preferred family of coordinate systems, in which the laws of special relativity hold locally, while the energy-momentum is conserved globally.

In the present paper the special coordinates, called "nonrotating," are analyzed further. First, a detailed proof of their existence is given. A way to construct such frames out of a given coordinate system is demonstrated, and the concept of "nonrotation" is elaborated. An explicit example of these frames is demonstrated for a wide range of curved manifolds. Particular attention is given to the relationship between the nonrotating frames and the Newton inertial frames in flat spacetime.

In the next section we will show the existence of this preferred class and in Section 3 its relation to the geodesic systems will be discussed. The integral form of the globally valid conservation law for the energy-momentum is given in Section 4. In Section 5 the relationship between the nonrotating frames and the Newton fixed-star system is discussed. Section 6 is devoted to concluding remarks, while the Appendix illustrates the existence of the geodesic nonrotating systems via a simple example.

2. THE NONROTATING CLASS OF COORDINATE SYSTEMS

The explicit form of the covariant divergencelessness condition for the energy-momentum tensor, equation (1), is

$$(\sqrt{-g}\ T^{\alpha\beta})_{,\beta} + \sqrt{-g}\ \Gamma^{\alpha}_{\beta\gamma} T^{\beta\gamma} = 0 \tag{4}$$

This equation differs from the continuity equation by the presence of the Γ term, which is generally interpreted as the manifestation of the interchange of energy-momentum between matter and gravitation. According to this viewpoint, various expressions for the gravitational energy have been proposed. All of them stem from the search for a general covariant continuity equation. In contrast, we suggest to interpret the term involving the connections as the expression of the mixing of the components due to the "rotation" of the coordinate system. Consequently, we define as "nonrotating" systems those frames for which this term vanishes. That is, the systems for which the following constraint holds:

$$\Gamma^{\alpha}_{\beta\gamma} T^{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} G^{\beta\gamma} = 0 \tag{5}$$

(where G is the Einstein tensor), will be called nonrotating.

This constraint implies a DeDonder-type condition

$$(\sqrt{-g}\ G^{\alpha\beta})_{,\beta} = 0$$

and leads to the following differential equation for the transformation functions to a nonrotating system $x'(x)$, where equation (2) becomes globally valid:

$$\Gamma^{\sigma}_{\alpha\beta} G^{\alpha\beta} \frac{\partial x'^{\gamma}}{\partial x^{\sigma}} - G^{\alpha\beta} \frac{\partial^2 x'^{\gamma}}{\partial x^{\alpha}\, \partial x^{\beta}} = 0 \tag{6}$$

Notice that this equation has the same form as the differential equation defining the harmonic coordinates, in terms of which the metric tensor density has a vanishing ordinary divergence. It is also worthwhile to observe that this equation is consistent with the Mach principle, since the nonrotating character of the system is determined by the distribution of matter throughout space. Consequently, in an empty region of space, equation (6) becomes an identity and the nonrotating coordinates are subject only to continuity conditions.

Since the first term of equation (6) vanishes in a nonrotating system, the internal group of this preferred class of coordinates is defined by the D'Alembert-type equation

$$G^{\alpha\beta}\frac{\partial^2 x'^{\gamma}}{\partial x^{\alpha}\,\partial x^{\beta}}=0 \tag{7}$$

Clearly, it includes the group of affine transformations, as should be expected on basic physical grounds.

Suppose that (x) is one of these preferred systems. To gain more physical insight, we can assume, as will be shown in the next section, that the frame (x) is locally geodesic, too. Now, perform a global Lorentz transformation, e.g., a "rotation" of the form

$$x'^0=\gamma(x^0-\beta x^1)$$
$$x'^1=\gamma(x^1-\beta x^0)$$
$$x'^2=x^2$$
$$x'^3=x^3$$

where β is constant and $\gamma=1/\sqrt{(1-\beta^2)}$. Equation (5) remains valid in the new local geodesic nonrotating system (x'). The components of the energy-momentum in the (x') system, however, are linear combinations of those in the (x) system. Hence, if the global Lorentz transformation is replaced by a time-dependent one (with β being a function of time), then, clearly, the energy-momentum components in this new "rotating" (x') system will be functions of time through β, and undergo mixing with the passage of time. A nonvanishing Γ term expressing this mixing should appear. This fact supports our assumption that the Γ term is related to the rotating state of the frame.

3. THE NONROTATING GEODESIC SYSTEMS

In this section we will see that among the geodesic systems with respect to a given line A there exists a subset of nonrotating systems.

Suppose that (x) is a geodesic system of coordinates with respect to a timelike geodesic line A, namely

$$\Gamma^{\gamma}_{A\alpha\beta}=0 \tag{8}$$

(the subscript A affixed to a variable denotes the restriction of the variable to the curve A). Let

$$x'^{\gamma} = x_A^{\gamma} + \left.\frac{\partial x'^{\gamma}}{\partial x^i}\right|_A x^i + \frac{1}{2}\left.\frac{\partial^2 x'^{\gamma}}{\partial x^i\,\partial x^j}\right|_A x^i x^j + \cdots \tag{9}$$

(Latin indices run from 1 to 3) be the Taylor expansion of a transformation function $x'(x)$ in the neighborhood of A, where we have taken $x_A'^{\gamma} = x_A^{\gamma}$ and $x_A^i = 0$ for convenience.

The coefficients in this series can now be determined so that (x') will be both geodesic and nonrotating. First notice that, due to equation (8), setting

$$\left.\frac{\partial x'^{\gamma}}{\partial x^{\alpha}}\right|_A = \delta_{\alpha}^{\gamma} \tag{10}$$

and

$$\left.\frac{\partial^2 x'^{\gamma}}{\partial x^i\,\partial x^j}\right|_A = 0 \tag{11}$$

is tantamount to solving equation (6) along the line A. This choice establishes the geodesic nature of the (x') system with respect to the line A. Now, taking the values of the successive derivatives of both sides of equation (6) along the line A, we get a set of algebraic equations, which have to be satisfied by the coefficients of the third- and higher-order terms for (x') to be a nonrotating system. While it is not difficult to write down the conditions explicitly, we will omit this step here, as it is done in detail in the subsequent alternative derivation [equations (12)–(17)]. Of course, the coefficients obtained thereby are functions of the local values of the geometric parameters of space-time along the world-line A.

Conversely, given a nonrotating but nongeodesic system (x), whose existence is guaranteed by the form of the differential equation (6), one applies a transformation of the internal group of the class of nonrotating systems, defined by equation (7), subordinate to the additional constraint

$$\left.\frac{\partial x'^{\gamma}}{\partial x^{\sigma}}\right|_A \Gamma_{A\alpha\beta}^{\sigma} = \left.\frac{\partial^2 x'^{\gamma}}{\partial x^{\alpha}\,\partial x^{\beta}}\right|_A \tag{12}$$

Then, according to the transformation law of the Christoffel symbols, the new coordinate system (x') will be nonrotating and geodesic with respect to the line A. The compatibility of condition (12) with equation (7) can be checked by observing that the contractions of both sides of equation (12) with the Einstein tensor G vanish as a consequence of equations (7) and (5), namely

$$G_A^{\alpha\beta} \left.\frac{\partial^2 x'^{\gamma}}{\partial x^{\alpha}\,\partial x^{\beta}}\right|_A = G_A^{\alpha\beta}\Gamma_{A\alpha\beta}^{\gamma} = 0 \tag{13}$$

To verify explicitly the existence of this transformation, let us write the transformation functions in the following form:

$$x'^\alpha = x^\alpha_A(x^0) + \frac{\partial x'^\alpha}{\partial x^i}\bigg|_A (x^0)x^i + \frac{1}{2}\frac{\partial^2 x'^\alpha}{\partial x^j\,\partial x^k}\bigg|_A (x^0)x^j x^k + F^\alpha_{ijk}(x)x^i x^j x^k \quad (14)$$

Where F^α_{ijk} replaces the terms of third and higher order of the Taylor expansion. First, the coefficients $(\partial x'^\alpha/\partial x_j)_A$ and $(\partial^2 x'^\alpha/\partial x^j\,\partial x^k)_A$ are required to satisfy equation (12), which entails the following two conditions:

$$\text{(a)}\quad \frac{d}{dx^0}\left[\frac{\partial x'^\alpha}{\partial x^j}\bigg|_A (x^0)\right] = \frac{\partial x'^\alpha}{\nabla x^\sigma}\bigg|_A (x^0)\Gamma^\sigma_{A0j}(x^0)$$

$$\text{(b)}\quad \frac{\partial^2 x'^\alpha}{\partial x^i\,\partial x^j}\bigg|_A (x^0) = \frac{\partial x'^\alpha}{\partial x^\sigma}\bigg|_A (x^0)\Gamma^\sigma_{Aij}(x^0) \quad (15)$$

For each α, equation (15a) constitutes a system of three linked ordinary first-order differential equations for the three functions $(\partial x'/\partial x)_A$, while (15b) determines the second-order coefficients. On the other hand, the requirement that the transformation $x'(x)$ belongs to the internal group, equation (7), imposes on the functions F the following condition [taking into account equations (15)]

$$G^{00}\left\{\frac{d}{dx^0}\left[\frac{\partial x'^\gamma}{\partial x^\sigma}\bigg|_A \Gamma^\sigma_{Ai0}\right]x^i + \frac{1}{2}\frac{d^2}{dx^{02}}\left[\frac{\partial x'^\gamma}{\partial x^\sigma}\bigg|_A \Gamma^\sigma_{Aij}\right]x^i x^j + \frac{\partial^2 F^\gamma_{ijk}}{\partial x^{02}}x^i x^j x^k\right\}$$

$$+ G^{0i}\left\{\frac{\partial x'^\gamma}{\partial x^\sigma}\bigg|_A \Gamma^\sigma_{Ai0} + \frac{d}{dx^0}\left[\frac{\partial x'^\gamma}{\partial x^\sigma}\bigg|_A \Gamma^\sigma_{Aij}\right]x^j + \frac{\partial^2 F^\gamma_{ljk}}{\partial x^0\,\partial x^i}x^l x^j x^k + 3\frac{\partial F^\gamma_{ijk}}{\partial x^0}x^j x^k\right\}$$

$$+ G^{ij}\left\{\frac{\partial x'^\gamma}{\partial x^\sigma}\bigg|_A \Gamma^\sigma_{Aij} + \frac{\partial^2 F^\gamma_{lmk}}{\partial x^i\,\partial x^j}x^l x^m x^k + 6\frac{\partial F^\gamma_{ljk}}{\partial x^i}x^l x^k + 6F^\gamma_{ijk}x^k\right\} = 0 \quad (16)$$

For each γ it is a second-order differential equation for the ten functions F, so that a great deal of freedom still remains.

Inquiring into the existence of analytic solutions, notice that since A is a geodesic line, i.e., $\Gamma^\sigma_{A00} = 0$, Eq. (16) can be written as

$$G^{\alpha\beta}(x)\Gamma^\sigma_{A\alpha\beta}(x^0)\frac{\partial x'^\gamma}{\partial x^\sigma}\bigg|_A (x^0) + x^j H^\gamma_{\alpha\beta j}(x)G^{\alpha\beta}(x) = 0 \quad (17)$$

The second term in equation (17) vanishes for $x^j = 0$. Consequently, the condition for the existence of analytic solutions for H is precisely equation (13); thereby the convergence of the Taylor expansion (14) is guaranteed in an appropriate neighborhood of the point or line A.

4. INTEGRAL CONSERVATION LAWS

In the usual manner we obtain from the ordinary divergencelessness condition (2), holding globally in nonrotating systems of coordinates, that

$$\frac{d}{dt}\int_{v}\sqrt{-g}\,T^{\alpha 0}\,dx=-\int_{\Sigma}\sqrt{-g}\,T^{\alpha i}\,d\sigma_i \tag{18}$$

where V is an arbitrary spatial volume with boundary Σ. Now if the volume integration is performed over an isolated system, so that the energy-momentum tensor vanishes over the surface boundary during the lapse of time from t_1 to t_2, we get that the volume integrals at the times t_1 and t_2 are *strictly* equal:

$$P^{\alpha}(t_1)=\int_{v}\sqrt{-g}\,T^{\alpha 0}\,dx\bigg|_{t_1}=\int_{v}\sqrt{-g}\,T^{\alpha 0}\,dx\bigg|_{t_2}=P^{\alpha}(t_2) \tag{19}$$

Furthermore, if the system of coordinates employed is in addition locally Lorentzian (the existence of such frames has been demonstrated above), these integrals give the values of the energy-momentum affine 4-vector as normally measured in the laboratory.

The existence of the nonrotating-geodesic systems, where equation (19) is valid, is a mathematical fact derived from the Einstein field equation without any additional assumption. On the other hand, the question of whether the conserved energy-momentum defined thereby does or does not include a gravitational contribution depends on the physical interpretation of T.

Note that, unlike the case of a nonrotating geodesic system, in an ordinary geodesic system of coordinates, equation (19) holds only approximately over a finite volume.

5. THE FLAT SPACETIME APPROXIMATION (OR THE NEWTON FIXED-STARS SYSTEM)

In order to find a connection between the concept of "nonrotating frame" as defined above and the conventional notion of nonrotating system in special relativity (Newton's "fixed-stars frame"), we now consider a special case similar to the example analyzed in the Appendix. Let us assume that the energy-momentum tensor is composed of: (a) a laboratory experimental setup such as oscillating small spherical masses attached to springs. The gravitational affect of this experimental setup will be neglected in our approximation procedure; and (b) massive stars distributed in space far away from our experimental setup, which will be considered as point masses.

Under these conditions the energy-momentum tensor will be expressed by

$$T^{\alpha\beta} = T_s^{\alpha\beta} + \frac{1}{\sqrt{-g}} \sum_i \prod_j \delta[x^j - x_i^j(x^0)] M_i U_i^\alpha(x^0) U_i^\beta(x^0) \tag{20}$$

where i runs over stars, j runs over the three spatial coordinates, x^0 is the temporal coordinate, and T_s denotes the energy-momentum tensor of the experimental setup.

To gain insight into the nature of the nonrotating systems under these conditions, consider first the zero approximation, where the T_s term is ignored entirely. Then the physical system under consideration reduces to a special case of the example studied in the Appendix. We may, therefore, make use of the result obtained there, namely, that a necessary and sufficient condition, in this case, for a coordinate system to be nonrotating is equation (A1),

$$U^\mu_{,\nu} U^\nu = 0$$

In the present case, however, U^μ depends on X^0 only, and the last condition becomes

$$U^\mu_{,0} = 0$$

which states that, in this configuration, a frame is nonrotating if and only if the stars are a rest, or in uniform motion, with respect to it.

Now let us take into account the laboratory system, which is assumed to be small in its spatial dimension and traces a certain geodesic path in spacetime. Denote the laboratory geodesic world-line by A, and consider again the zero approximation and a nonrotating frame S identified there. In view of the general construction presented in Section 3, there exists a coordinate transformation—member of the internal group of the class of nonrotating systems—which transforms the nonrotating frame S into another nonrotating frame S' which is geodesic with respect to the line A. It is now easy to see that S' is a nonrotating frame for the physical situation under consideration, namely, with the full T given by equation (20), for in the zone of the faraway stars the two situations, with and without T_a, are indistinguishable, while in the neighborhood of the geodesic line A (in our approximation T_s does not affect the geometry of spacetime, so that the geodesic lines in the two situations are identical) the energy-momentum tensor satisfies approximately an ordinary divergenceless condition. Hence in the S' frame T satisfies globally

$$(\sqrt{-g}\; T^{\alpha\beta})_{,\beta} = 0$$

which defines S' as a nonrotating frame.

In conclusion, the geodesic-nonrotating frames introduced in this paper are in the flat-space approximation none other than Newton's fixed-stars systems, whereby locally Newton's laws of motion are valid (geodesic frames) and the stars are at rest or in uniform motion.

6. CONCLUSION AND REMARKS

The problem of conservation of energy in general relativity has been discussed based on a coordinate approach.

It has been demonstrated that there exists a class of preferred coordinate systems, the nonrotating frames, in which an ordinary continuity equation for the energy-momentum tensor holds globally. These preferred frames are determined exclusively by the distribution of energy-momentum throughout space-time.

The existence of this preferred class of frames entails the conservation of spatial integrals of the energy-momentum. Furthermore, for a given observer tracing a timelike geodesic curve in space-time, a subset of the class of nonrotating frames will be geodesic with respect to that observer. Measured in these coordinate systems, the conserved integrals correspond to the physical quantities interpreted as the total energy-momentum of the system under consideration. Thus, with the aid of the nonrotating frames, the conservation of energy is established in a physically meaningful manner, with no need to introduce an extraneous pseudotensor. The global conservation of the energy-momentum tensor established in the present paper is an indication that this tensor represents the total energy-momentum, including the gravitational contribution.

Finally, it is worth stressing that the existence of a preferred class of coordinates is an experimental fact. Since the Newton water bucket clearly distinguishes between rotating and nonrotating frames, it is obvious that in special relativity the energy-momentum is conserved only in these physically preferred "nonrotating" frames selected by Newton's bucket. It should, therefore, come as no surprise that in curved spacetime, too, the energy-momentum is conserved only in a preferred class of frames.

APPENDIX: AN EXAMPLE

As a concrete realization of the foregoing discussion, consider the special case of a dust cloud characterized by a proper density $\rho(x)$ and 4-velocity $u^{\mu}(x)$ in a region of space-time with any arbitrary metric compatible with the presence of the dust. The underlying energy-momentum tensor is

$$T^{\mu\nu} = \rho u^{\mu} u^{\nu}$$

where ρ is a scalar and u^μ is a unit vector. Notice that any possible direct contribution of the gravitational energy to T is neglected (compare with the remark in Section 6).

The divergencelessness condition on $T^{\mu\nu}$ implies

$$\rho u^\mu_{;\nu} u^\nu + u^\mu (\rho u^\nu)_{;\nu} = 0$$

which, upon contraction with u_μ, yields

$$(\rho u^\nu)_{;\nu} = 0$$

Substituting back, one finds

$$u^\mu_{;\nu} u^\nu = 0$$

or

$$u^\mu_{,\nu} u^\nu + \rho^{-1} \Gamma^\mu_{\nu\gamma} T^{\nu\gamma} = 0$$

Hence, a coordinate system in the situation under consideration is a nonrotating frame if and only if the 4-velocity u^μ satisfies

$$u^\mu_{,\nu} u^\nu = 0 \tag{A1}$$

In particular, a "comoving frame" defined by

$$u^\mu = \delta^\mu_0$$

is necessarily nonrotating. Notice that in such frames one has

$$\Gamma^\mu_{00} = 0$$

In view of equation (7) for the internal group, the most general nonrotating system x' is obtained from the comoving frame x by the transformation

$$x'^\mu = x^0 A^\mu(x^k) + B^\mu(x^k) \tag{A2}$$

where A^μ and B^μ are arbitrary functions of the coordinates x^1, x^2, x^3. For x' to be in addition a geodesic system with respect to a given point or line A, the functions A^μ and B^μ must be subordinate to the constraint (12), viz.,

$$A^\mu_{,k}|_A = \Gamma^\mu_{A0k}$$

$$x^0 A^\mu_{,kj} + B^\mu_{,kj}|_A = \Gamma^\mu_{Akj}$$

[equation (12) for $\alpha = \beta = 0$ is satisfied identically].

It is evident, then, that by an appropriate choice of the functions A^μ and B^μ, the transformation (A2), applied to a comoving frame, yields a nonrotating system geodesic with respect to the given point or line.

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